

# A Hamiltonian Formalism for Dissipative Mechanical Systems

Tianshu Luo, Yimu Guo

*Institute of Solid Mechanics, Department of Applied Mechanics, Zhejiang University,  
Hangzhou, Zhejiang, 310027, P.R.China*

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## Abstract

In this paper an approach is proposed to introduce Hamiltonian formalism to dissipative mechanical systems. This approach is based upon below viewpoints proposed in this paper: under a certain identical initial condition a dissipative mechanical system shares a common phase flow curve with a conservative system; the Hamiltonian of the conservative system is the value of the total energy of the dissipative system under the initial condition; whether the dissipative mechanical system has an analytical solution or not, the conservative system may always exist, which is to say the existence of the Hamiltonian and Hamilton's equation. Hence this approach is to substitute an infinite number of conservative systems for a dissipative mechanical system corresponding to varied initial conditions. The major means to demonstrate these viewpoints is that by the Newton-Laplace principle the nonconservative force can be reasonably assumed as a function of a component of generalized coordinates  $q_i$  along. The advantage of this approach is such that there is no need to change the definition of canonical momenta and the motion is identical as that of the original system; this approach is suitable to all dissipative mechanical systems regardless of the existence of analytical solution.

*Key words:* Hamiltonian formalism, dissipation, non-conservative system, damping

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*Email address:* ltsmechanic@zju.edu.cn, guoyimu@zju.edu.cn (Tianshu Luo, Yimu Guo).

## 1 Introduction

Since Hamilton originated Hamilton equations of motion and Hamiltonian formalism, it has been stated in most classical textbooks that the Hamiltonian formalism focuses on solving conservative problems.

In 1960s, Hori and Brouwer (1961) utilized the classical Hamiltonian formalism and a perturbation theory to solve a non-conservative problem. They did not attempt to derive the Hamiltonian formalism for non-conservative problems. In 1990s Tveter (1998) started an attempt and thereby obtained his general Hamilton equation:

$$\begin{aligned}\dot{P}_i &= - \left( \frac{\partial K}{\partial Q_i} \right)_{QPt} + \mathbf{F} \left( \frac{\partial r}{\partial Q_i} \right)_{QPt} \\ \dot{Q}_i &= \left( \frac{\partial K}{\partial P_i} \right)_{QPt} - \mathbf{F} \left( \frac{\partial r}{\partial P_i} \right)_{QPt},\end{aligned}\tag{1.1}$$

where  $\{Q, P\}$  denote the canonical variables and the position vector  $r$  depends on the canonical variable set  $\{Q, P\}$  and  $t$ , i.e.  $r(Q, P, t)$ ,  $K$  is the transformed Hamiltonian and the subscript in the partial derivative expressions indicate the functional dependencies of  $K$  and  $r$ . If the variable set  $\{Q, P\}$  is transformed into the variable set  $\{q, p\}$ , where  $p$  is mechanical momentum, the position vector  $r$  depends on  $q$  and does not depend explicitly on  $t$ , i.e.,  $r(q)$ , Eq. (1.1) could be reduced to

$$\begin{aligned}\dot{p}_i &= - \left( \frac{\partial H}{\partial q_i} \right)_{qp} + \mathbf{F} \left( \frac{\partial r}{\partial q_i} \right)_q \\ \dot{q}_i &= \left( \frac{\partial H}{\partial p_i} \right)_{qp},\end{aligned}\tag{1.2}$$

where  $\mathbf{F}(\partial r / \partial q_i)_q$  denotes a generalized force in direction  $i$  and  $H$  is the Hamiltonian. Yet both Eq. (1.1) and Eq. (1.2) are not truly Hamiltonian formalisms, because there is not a conservative Hamiltonian quantity (first integral). In 1990s other similar attempts were launched, but the resulting formalisms were not truly Hamiltonian.

In 1940s P.Morse and Feshbach (1953) gave an example of an artificial Hamiltonian for a damped oscillator based on a “mirror-image” trick, incorporating a second oscillator with negative friction. The resulting Hamiltonian is un-physical: it is unbounded from below and under time reversal the oscillator is transformed into its “mirror-image”. By this arbitrary trick dissipative systems can be handled as though they were conservative. Vujanovic

(1989) investigated dissipative systems from the vision of variational methods. Tarasov (2001) suggested a generalization of canonical quantization that maps a dynamical operator to a dynamical superoperator. Tarasov (2001) claimed that this approach allows defining consistent quantization procedure for non-Hamiltonian and dissipative systems. R.M.Kiehn (1974) considered that dissipation effects may be included by considering the dissipative systems for which the closed integral of action is a parameter-dependent, conformal invariant of the motion. He applied this idea to hydrodynamics. Rajeev (2007) considered that a large class of dissipative systems can be brought to a canonical form by introducing complex coordinates in phase space and a complex-valued Hamiltonian. The papers all took into consideration, the energy drained from the dissipative system. In general they used an example of a damped oscillator to demonstrate their approach. But the equations describing the oscillator always were one-dimensional or decoupled to a group of one-dimensional equations. For instance, Vujanovic (1989) offered an example : for the linear damped oscillator  $m\ddot{x} + \mu\dot{x} + cx = 0$ , the following conservation law can be obtained:  $((\dot{x}^2/2) + (\omega^2 x^2/2) + kx\dot{x})e^{2kt} = \text{const}$ , where  $\omega^2 = c/m$  and  $2k = \mu/m$ . We can see that  $e^{2kt}$  is a decay coefficient of amplitude. We think, if and only if the linear damped oscillator is one dimensional or can be decoupled to a group of one-dimensional equations, the decay coefficient can be presented as  $e^{2kt}$ . In addition we consider, that if one were to apply to these approaches to dissipative systems, one must first change the definition of canonical momenta and canonical coordinates.

Based on Tveter's Eq.(1.2), in this paper we attempt to apply Hamilton formalism to mechanical systems with damping forces. The approach is to substitute a group of conservative systems for a dissipative mechanical system. We have found that one can take the value of the total energy of the dissipative system as the Hamiltonian of a conservative system, the motion of which is the same as that of the dissipative system with a certain initial condition.

## 2 The derivation

### 2.1 Obtaining the Hamiltonian Quantity of the substituting conservative system

Under general circumstances, the force  $\mathbf{F}$  is a damping force that depends on the variable set  $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ . We denote by  $F_i$  the components of the generalized force  $\mathbf{F}$ .

$$F_i(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) = \mathbf{F} \left( \frac{\partial r}{\partial q_i} \right). \quad (2.1)$$

Thus we can reformulate the Eq.(1.2) as follows:

$$\begin{aligned}\dot{p}_i &= - \left( \frac{\partial H}{\partial q_i} \right)_{qp} + F_i(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) \\ \dot{q}_i &= \left( \frac{\partial H}{\partial p_i} \right)_{qp}.\end{aligned}\tag{2.2}$$

Suppose the new Hamiltonian Quantity is  $\hat{H}$  and we do not change the definition of the canonical momenta, thus we may write a Hamilton's equation of a new conservative system:

$$\begin{aligned}\dot{\hat{p}}_i &= - \left( \frac{\partial \hat{H}}{\partial \hat{q}_i} \right)_{\hat{q}\hat{p}} \\ \dot{\hat{q}}_i &= \left( \frac{\partial \hat{H}}{\partial \hat{p}_i} \right)_{\hat{q}\hat{p}},\end{aligned}\tag{2.3}$$

where the initial condition is same as that of Eq.(2.2), i.e.  $\hat{p}_0 = p_0, \hat{q}_0 = q_0$ . Under the initial condition described as above, we assume  $\hat{p} = p, \hat{q} = q$ . By comparing Eq.(2.2) with Eq.(2.3), we have

$$\begin{aligned}\left( \frac{\partial \hat{H}}{\partial \hat{q}_i} \right)_{\hat{q}\hat{p}} &= \left( \frac{\partial H}{\partial q_i} \right)_{qp} - F_i(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) \\ \left( \frac{\partial \hat{H}}{\partial \hat{p}_i} \right)_{\hat{q}\hat{p}} &= \left( \frac{\partial H}{\partial p_i} \right)_{qp}.\end{aligned}\tag{2.4}$$

In classical mechanics the Hamiltonian  $H$  is mechanical energy:

$$H = \int_{\gamma} \left( \frac{\partial H}{\partial q_i} \right)_{qp} dq_i + \int_{\gamma} \left( \frac{\partial H}{\partial p_i} \right)_{qp} dp_i + c_1,\tag{2.5}$$

where  $\gamma$  denotes a phase flow curve presented by Eq.(2.2),  $c_1$  is a constant that depends on the initial condition above. The Einstein summation convention has been used in this paper.

Hence an attempt is made to find  $\hat{H}$  through line integral along the curve  $\gamma$ :

$$\int_{\gamma} \left( \frac{\partial \hat{H}}{\partial \hat{q}_i} \right)_{\hat{q}\hat{p}} d\hat{q}_i = \int_{\gamma} \left[ \left( \frac{\partial H}{\partial q_i} \right)_{qp} - F_i(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) \right] d\hat{q}_i$$

$$\int_{\gamma} \left( \frac{\partial \hat{H}}{\partial \hat{p}_i} \right)_{\hat{q}\hat{p}} d\hat{p}_i = \int_{\gamma} \left( \frac{\partial H}{\partial p_i} \right)_{qp} dp_i. \quad (2.6)$$

In like manner with Eq.(2.5), we have

$$\hat{H} = \int_{\gamma} \left( \frac{\partial \hat{H}}{\partial \hat{q}_i} \right)_{\hat{q}\hat{p}} d\hat{q}_i + \int_{\gamma} \left( \frac{\partial \hat{H}}{\partial \hat{p}_i} \right)_{\hat{q}\hat{p}} d\hat{p}_i + c_2, \quad (2.7)$$

where  $c_2$  is a constant which depends on the initial condition. On the path  $\gamma$  we have

$$d\hat{q}_i = dq_i, d\hat{p}_i = dp_i \quad (2.8)$$

Substituting Eq.(2.5)(2.6)(2.8) into Eq.(2.7), it can be represented as:

$$\hat{H} = H - \int_{\gamma} F_i(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dq_i + c. \quad (2.9)$$

where  $c = c_2 - c_1$ . According to the physical meaning of Hamiltonian,  $c_1$  and  $c_2$  and  $c$  are added into Eq.(2.5)(2.6)(2.9) such that the integral constant vanishes in the Hamiltonian quantity. It is well known that Eq.(2.2) denotes a phase flow curve, thus according to the Newton-Laplace principle of determinacy described in book written by V.I.Arnold (1978), we can assume

$$\begin{aligned} q_i &= q_i(t) \\ \dot{q}_i &= \dot{q}_i(t), \end{aligned}$$

where the flow curve satisfies the initial condition. Hence we divide time domain into a group of sufficient small domain and consider a small domain, such that we can reasonably assume  $F_i$  as

$$F_i(q_1(t(q_i)), \dots, q_n(t(q_i)), \dot{q}_1(t(q_i)), \dots, \dot{q}_n(t(q_i))) = \mathcal{F}_i(q_i),$$

where  $\mathcal{F}_i$  is a function of  $q_i$  alone. Indeed  $\mathcal{F}$  is a conservative force, the value of that is equal to the value of  $F_i$  at  $\gamma$ . Thus we have

$$\int_{\gamma} F_i dq_i = \int_{q_i^0}^{q_i} \mathcal{F}_i(q_i) dq_i = W_i(q_i) - W_i(q_i^0). \quad (2.10)$$

According to the physical meaning of Hamiltonian,  $c$  is added to Eq.(2.9) such that the integral constant vanishes in Hamiltonian quantity. Hence  $c = -W_i(q_i^0)$ . Substituting Eq.(2.10) and  $c = -W_i(q_i^0)$  into Eq.(2.9), we have

$$\hat{H} = H - W_i(q_i) \quad (2.11)$$

where  $-W_i(q_i)$  denotes the negative work done by the damping force  $F$ . In Eq.(2.11)  $\hat{H}$  and  $H$  are both functions of  $q_i$  and  $W_i(q_i)$  a function of  $q_i$ . If we substitute  $\hat{q}_i$  and  $\hat{p}_i$  for  $q_i$  and  $p_i$ , and do not change the form of functions  $\hat{H}$  and  $H$ ,  $W_i(q_i)$ , we would then obtain the Hamiltonian of Eq.(2.3)

$$\hat{H}(\hat{q}_i, \hat{p}_i) = H(\hat{q}_i, \hat{p}_i) W_i(\hat{q}_i) \quad (2.12)$$

Then we must show that the Hamiltonian presented by Eq.(2.12) satisfies the Eq.(2.4) under the same initial condition, i.e. that the motion of Eq.(2.2) is equivalent to that of Eq.(2.3) under the same initial condition.

Substituting Eq.(2.12) into the left side of Eq.(2.4), we have

$$\begin{aligned} \frac{\partial \hat{H}(\hat{q}_i, \hat{p}_i)}{\partial \hat{q}_i} &= \frac{\partial H(\hat{q}_i, \hat{p}_i)}{\partial \hat{q}_i} - \frac{\partial W_j(\hat{q}_j)}{\partial \hat{q}_i} \\ \frac{\partial \hat{H}(\hat{q}_i, \hat{p}_i)}{\partial \hat{p}_i} &= \frac{\partial H(\hat{q}_i, \hat{p}_i)}{\partial \hat{p}_i} - \frac{\partial W_j(\hat{q}_j)}{\partial \hat{p}_i}. \end{aligned} \quad (2.13)$$

It must be emphasized that although  $q_i$  and  $p_i$  are considered as distinct variables in Hamilton's mechanics, we can consider  $q_i$  and  $\dot{q}_i$  as dependent variables in the process of constructing of  $\hat{H}$ . At the trajectory  $\gamma$  we have

$$\begin{aligned} \frac{\partial W_j(q_j)}{\partial \hat{q}_i} &= \frac{\partial (\int_{\hat{q}_j^0}^{\hat{q}_j} \mathcal{F}_j(\hat{q}_j) d\hat{q}_j)}{\partial \hat{q}_i} = \mathcal{F}_i(\hat{q}_i) \\ \frac{\partial W_j(q_j)}{\partial \hat{p}_i} &= 0, \end{aligned} \quad (2.14)$$

where  $\mathcal{F}_i(\hat{q}_i)$  is the value of damping force  $F_i$  on the phase flow curve  $\gamma$ . Substituting Eq.(2.14) into Eq.(2.13) and substituting  $q_i$  and  $p_i$  for  $\hat{q}_i$  and  $\hat{p}_i$ , Eq.(2.13) can be reduced to Eq.(2.4) under the initial condition.

Therefore we can say that the motion of Eq.(2.3) is equivalent to that of Eq.(2.2) at the initial condition; the dissipative system presented by Eq.(2.2) can be replaced by the conservative system presented by Eq.(2.2) at the initial condition; and  $\hat{H}$  is the Hamiltonian of a substituting conservative system.

## 2.2 Example

Take an oscillator with a spring and damping as an example, the governing equation of which is as below:

$$\ddot{q} + c\dot{q} + kq = 0, \quad (2.15)$$

where the stiffness coefficient  $k$  is a constant, and the damping coefficient  $c$  also is a constant. The solution of Eq.(2.15) is

$$q = Ae^{-ct/2} \cos\left(\frac{(c^2 - 4k)^{\frac{1}{2}}}{2}t + B\right), \quad (2.16)$$

$$\begin{aligned} \dot{q} = & -A(c^2 - 4k)e^{-ct/2} \sin\left(\frac{(c^2 - 4k)^{\frac{1}{2}}}{2}t + B\right) \\ & - A \cdot \frac{c}{2}e^{-ct/2} \cos\left(\frac{(c^2 - 4k)^{\frac{1}{2}}}{2}t + B\right) \end{aligned} \quad (2.17)$$

where constants  $A$  and  $B$  are determined by an initial condition. We assume  $t$  belongs to a sufficient small domain, so we let

$$t = t(q) \quad (2.18)$$

As the section above by substituting Eq.(2.18) into Eq.(2.17) one can represent the damping force as a function of  $q$  alone:

$$c\dot{q} = c\dot{q}(q), \quad (2.19)$$

which can be rewritten as:

$$c\dot{q} = \tilde{k}(t)q. \quad (2.20)$$

Hence one obtains a new conservative system as below:

$$\ddot{\hat{q}} + (k + \tilde{k}(t))\hat{q} = 0. \quad (2.21)$$

This is a nonlinear oscillator, the stiffness of which varies with the time  $t$ . In order to make it easy to distinguish Eq.(2.21) from Eq.(2.15), we denote the position variable in Eq.(2.21) as  $\hat{q}$ . If and only if the Equation above has the same initial condition as Eq.(2.15), Eq.(2.21) shares the same solution with

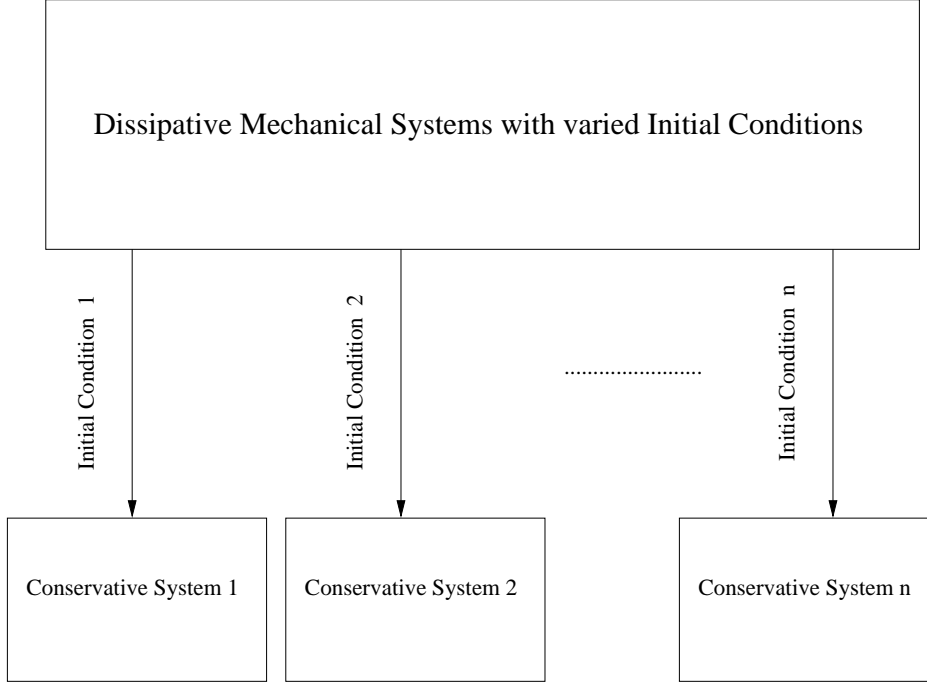


Fig. 1.

Dissipative Mechanical System and Conservative Systems

Eq.(2.15). The Hamiltonian quantity of Eq.(2.21) is the value of the total energy of the Eq.(2.15) under the initial condition:

$$\hat{H} = \frac{1}{2}\dot{q}^2 + \frac{1}{2}kq^2 + \int_{\gamma} c\dot{q}dq. \quad (2.22)$$

According to Eq.(2.19,2.20), the Hamiltonian can be represented as:

$$\hat{H} = \frac{1}{2}\dot{q}^2 + \frac{1}{2}kq^2 + \int_{\gamma} \tilde{k}(t)q\dot{q}dq = \frac{1}{2}\dot{\hat{q}}^2 + \frac{1}{2}k\hat{q}^2 + \int_{\gamma} \tilde{k}(t)\hat{q}d\hat{q} \quad (2.23)$$

It must be emphasized that  $\int_{\gamma} \tilde{k}(t)\hat{q}d\hat{q}$  is a function of  $\hat{q}$  alone, and  $\tilde{k}(t)$  is determined by the initial condition of Eq.(2.15). As such the Hamiltonian quantity  $\hat{H}$  is determined by the initial condition of Eq.(2.15).

### 2.3 Discussion

Based on the above, we can outline the relation between a dissipative mechanical system and conservative systems with Fig(1). As shown in Fig(1), the relation between a dissipative mechanical system and conservative systems can be stated as below:



The initial condition of the original dissipative system determines the phase flow curve  $\gamma$ ; the value of the total energy of the original system  $\hat{H}$  depends on  $\gamma$ ; one can take  $\hat{H}$  as the Hamiltonian of a substituting conservative system; the substituting system shares only one common phase flow curve  $\gamma$  with the original system.

Although the substituting system shares a common phase flow curve with the original system, under other initial conditions they exhibit different phase flow curves. Therefore the phase flow of the substituting systems differs from that of the original system. It is well known that the Liouville theorem as explained by V.I. Arnold (1978), can be formulated as:

**Theorem 2.1** *The phase flow of Hamilton's Equations preserves volume: for any region  $D$  we have*

$$\text{volume of } g^t D = \text{volume of } D$$

where  $g^t$  is the one-parameter group of transformations of phase space

$$g^t : (p(0), q(0)) \longmapsto (p(t), q(t))$$

According to the theorem above, the phase flow of the original dissipative system Eq.(2.2) certainly does not preserve its phase volume, but the phase flow of the substituting conservative Eq.(2.3) does.

The Hamiltonians of new conservative systems in general are not analytically integrable, unless the original mechanical system is integrable. The reason is that the work done by damping force depends on the phase flow curve. If the system is integrable, then the phase flow curve can be explicitly written out, the system has an analytical solution, and therefore the work done by damping force can be explicitly integrated. Therefore the Hamiltonian  $\hat{H}$  can be explicitly represented. Most systems do not have an analytical solution. Despite this, the Hamilton quantity and coordinates and momentum must satisfy the Eq.(2.3) under a certain initial condition. Why had F.Klein (1928) written: "Physicists can make use of these theories only very little, an engineers nothing at all"? The answer: when one is seeking an analytical solution to a classical mechanics problem by utilizing Hamiltonian formalism, in fact one must inevitably convert the problem back to Newtonian formalism. This means that an explicit form of Hamiltonian quantity is not necessary for classical mechanics. What is important is the relation between  $q, p$  and Hamiltonian quantity embodied in the Hamilton's Equation.

### 3 Conclusion

We can conclude that a dissipative mechanical system has a property: under a certain common initial condition the dissipative system shares a common phase flow curve with a conservative system; the Hamiltonian of the conservative system is the value of the total energy of the dissipative system. Based on this point it can be concluded, that a dissipative problem can be reformulated as an infinite number of non-dissipative problems, corresponding to every phase flow curve of the dissipative problem. One can avoid having to change the definition of the canonical momentum in the Hamilton formalism, because under a certain initial condition the motion of one of the group of conservative systems is same as the original dissipative system.

Because conservative systems may own some other fine characteristics (e.g. capable of preserving symplectic structure), it is feasible to substitute a group of conservative systems for the original dissipative mechanical system, and thereby introduce the Hamiltonian formalism to dissipative mechanical systems.

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